Mathematical Sciences Society Math Contest-2020

University of Alberta

(proposed by Muhammad Abul Fazal)

- *The Exam has 5 problems, you have 2 hours to complete and submit them.*

1. Let $A_n \in \mathbb{R}^{n\times n}$ be a matrix, such that $a_{ij} = 0$, if $|i-j| \geq 2$ and $a_{ii} = 1$, $a_{i+1,i} = -1$ and $a_{i,i+1} = 1$, Show that $\det(A_n)$ satisfies Fibonacci relation, i.e. $\det(A_n) = \det(A_{n-1}) + \det(A_{n-2})$ (10 points)

2. Let $h: [0,1] \to \mathbb{R}$ be a continuous function with the following properties:

a. h is differentiable on open interval $(0,1)$

b. $h(0) = h(1) = 0$

Prove that $g(x) = h(x) + 2h'(x)$ has atleast one zero on interval $(0,1)$. (10 points)

3. Prove that for $x, y \in \mathbb{Z}$, the equation:

$$x^3 + (x+1)^3 + (x+2)^3 + (x+3)^3 + (x+4)^3 + (x+5)^3 + (x+6)^3 = y^4 + (y+1)^4$$

Has no solutions. (10 points)

4. An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of same colour as that of ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k. (10 points)

5. Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{(1+(\tan x)^n)}$ for any $n \in \mathbb{R}$. (10 points)